Griffiths Problem 6.4

The infinitesimal current loop of side $\epsilon$ is oriented in the $y-z$ plane from $(0, 0, 0)$ to $(0, \epsilon, 0)$ then to $(0, \epsilon, \epsilon)$, on to $(0, 0, \epsilon)$ and back to the origin. Please see Figure 6.8 in the text. We compute the total force by summing (vectorially) the forces on each of the four segments using $d\vec{F} = Id\vec{\ell} \times \vec{B}$. The total force on the segments oriented along the $y$–axis is

$$\int_0^\epsilon \left[ I dy \hat{y} \times \vec{B}(0, y, 0) - I dy \hat{y} \times \vec{B}(0, y, \epsilon) \right].$$  

(1)

We use Taylor series expansion (in the variable $z$ expanding around $z = 0$) to write

$$\vec{B}(0, y, \epsilon) = \vec{B}(0, y, 0) + \epsilon \left. \frac{\partial \vec{B}}{\partial z} \right|_{(0, y, 0)}$$

noting that this constitutes three equations, one for each component. We can simplify this by approximating the partial derivative as follows:

$$\left. \frac{\partial \vec{B}}{\partial z} \right|_{(0, y, 0)} \approx \left. \frac{\partial \vec{B}}{\partial y} \right|_{(0, 0, 0)}$$

to the order in $\epsilon$ we wish to retain. Substituting this into Equation (1) and integrating we obtain

$$-I \epsilon^2 \hat{y} \times \left. \frac{\partial \vec{B}}{\partial z} \right|_{(0, 0, 0)}.$$

(2)

Similarly, the segments oriented along $z$ yield

$$\int_0^\epsilon \left[ I dz \hat{z} \times \vec{B}(0, \epsilon, z) - I dz \hat{z} \times \vec{B}(0, 0, z) \right]$$

$$\approx \int_0^\epsilon \left[ I dz \hat{z} \times (\vec{B}(0, 0, z) + \epsilon \left. \frac{\partial \vec{B}}{\partial y} \right|_{(0, 0, 0)} - \vec{B}(0, 0, z)) \right]$$

$$= I \epsilon^2 \hat{z} \times \left. \frac{\partial \vec{B}}{\partial y} \right|_{(0, 0, 0)}$$

(3)

We can write $I \epsilon^2 = m$ the magnetic moment since it is the current times the area; it points along $+\hat{x}$. We now evaluate the cross products and simplify:

$$\hat{z} \times \left. \frac{\partial \vec{B}}{\partial y} \right|_{(0, 0, 0)} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial B_x}{\partial y} & \frac{\partial B_y}{\partial y} & \frac{\partial B_z}{\partial y} \end{vmatrix} = -\hat{x} \frac{\partial B_y}{\partial y} + \hat{y} \frac{\partial B_z}{\partial y}.$$  

The other term yields

$$-\hat{y} \times \left. \frac{\partial \vec{B}}{\partial z} \right|_{(0, 0, 0)} = -\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial B_x}{\partial z} & \frac{\partial B_y}{\partial z} & \frac{\partial B_z}{\partial z} \end{vmatrix} = -\hat{x} \frac{\partial B_z}{\partial z} + \hat{z} \frac{\partial B_x}{\partial z}.$$  

Assembling all the terms together the force is given by

$$\vec{F} = m \left[ -\hat{x} \left( \frac{\partial B_z}{\partial z} + \frac{\partial B_y}{\partial y} \right) + \hat{y} \frac{\partial B_x}{\partial y} + \hat{z} \frac{\partial B_x}{\partial z} \right].$$
Now using $\mathbf{\nabla} \cdot \vec{B} = 0$ we can re-write the $x$-component\(^1\) and obtain
\[
\vec{F} = m \left( \hat{x} \frac{\partial B_x}{\partial x} + \hat{y} \frac{\partial B_x}{\partial y} + \hat{z} \frac{\partial B_x}{\partial z} \right).
\]

Since $\vec{m} = m\hat{x}$ we note that
\[
\mathbf{\nabla} \left( \vec{m} \cdot \vec{B} \right) = \mathbf{\nabla} (mB_x) = m \left( \hat{x} \frac{\partial B_x}{\partial x} + \hat{y} \frac{\partial B_x}{\partial y} + \hat{z} \frac{\partial B_x}{\partial z} \right)
\]
which is identical to the expression above and so we have $\vec{F} = \mathbf{\nabla} (\vec{m} \cdot \vec{B})$. You must remember this!

\(^1\) $\frac{\partial B_x}{\partial x} = -\frac{\partial B_y}{\partial y} - \frac{\partial B_z}{\partial z}$.